

ON NON-KÄHLER DEGREES OF COMPLEX MANIFOLDS

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ABSTRACT. We study cohomological properties of complex manifolds. In particular, under suitable metric conditions, we extend to higher dimensions a result by A. Teleman, which provides an upper bound for the Bott-Chern cohomology in terms of Betti numbers for compact complex surfaces according to the dichotomy b_1 even or odd.

INTRODUCTION

Let X be a compact complex manifold. Consider the double complex $(\wedge^{\bullet,\bullet} X, \partial, \bar{\partial})$.

Besides de Rham and Dolbeault cohomology, consider the *Bott-Chern* [12] and *Aeppli* [1] cohomologies, defined as:

$$H_{BC}^{\bullet,\bullet}(X) := \frac{\ker \partial \cap \ker \bar{\partial}}{\operatorname{im} \partial \bar{\partial}} \quad \text{and} \quad H_A^{\bullet,\bullet} := \frac{\ker \partial \bar{\partial}}{\operatorname{im} \partial + \operatorname{im} \bar{\partial}}.$$

The Hodge theory developed by M. Schweitzer [26] assures their finite-dimensionality as \mathbb{C} -vector spaces. The identity induces the natural maps

$$\begin{array}{ccccc} & & H_{BC}^{\bullet,\bullet}(X) & & \\ & \swarrow & \downarrow & \searrow & \\ H_{\partial}^{\bullet,\bullet}(X) & & H_{dR}^{\bullet}(X; \mathbb{C}) & & H_{\bar{\partial}}^{\bullet,\bullet}(X) \\ & \searrow & \downarrow & \swarrow & \\ & & H_A^{\bullet,\bullet}(X) & & \end{array}$$

of (bi-)graded vector spaces. One says that X satisfies the $\partial\bar{\partial}$ -Lemma if the natural map $H_{BC}^{\bullet,\bullet}(X) \rightarrow H_{dR}^{\bullet}(X; \mathbb{C})$ is injective. This turns out to be equivalent to all the above maps being isomorphisms, [15, Lemma 5.15, 5.21, Remark 5.16]. Therefore, while compact Kähler manifolds satisfy the $\partial\bar{\partial}$ -Lemma, for complex non-Kähler manifolds the above maps may be neither injective nor surjective. A (non-canonical) comparison between Bott-Chern and Aeppli cohomologies and de

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Rham cohomology is provided by the inequality *à la* Frölicher in [8, Theorem A]. More precisely, for any $k \in \mathbb{Z}$, we have the non-negative degree

$$\Delta^k := \dim_{\mathbb{C}} H_{BC}^k(X) + \dim_{\mathbb{C}} H_A^k(X) - 2b_k \in \mathbb{N},$$

where $H_{BC}^k(X) := \bigoplus_{p+q=k} H_{BC}^{p,q}(X)$, (and same for Aeppli,) and $b_k := \dim_{\mathbb{C}} H_{dR}^k(X; \mathbb{C})$ denotes the k th Betti number. The validity of the $\partial\bar{\partial}$ -Lemma is characterized by $\Delta^k = 0$ for any $k \in \mathbb{Z}$, [8, Theorem B]. Such a result is extended to generalized-complex structures, here including symplectic structures, in [9, 14]. An upper-bound of the dimensions of the Bott-Chern cohomology in terms of the Hodge numbers is provided in [6].

In this note, we study the cohomology of compact complex manifolds.

In general, the degrees Δ^k measure the failure of $\partial\bar{\partial}$ -Lemma, that is, non-cohomologically-Kählerness. In fact, they measure non-Kählerness for compact complex surfaces. This is because of the topological characterization of Kählerness in terms of the parity of the first Betti number, [22, 25, 27], see also [23, Corollaire 5.7], and [13, Theorem 11].

In [29], (see also [24],) A. Teleman proves that, for compact complex surfaces, Δ^1 is always zero and $\Delta^2 \in \{0, 2\}$; and, in [3], the Bott-Chern and Aeppli cohomologies for compact complex surfaces diffeomorphic to solvmanifolds and for class VII surfaces are computed.

Theorem 1.1 ([29, Lemma 2.3]). *Let X be a compact complex surface. Then:*

- b_1 is even if and only if $\Delta^1 = \Delta^2 = 0$;
- b_1 is odd if and only if $\Delta^1 = 0$ and $\Delta^2 = 2$.

In particular, it follows that $\Delta^1 \in \{0\}$ and $\Delta^2 \in \{0, 2\}$ are topological invariants of compact complex surfaces. This is no more true in higher dimension, see, e.g., the examples in [2] on the Iwasawa manifold and its small deformations.

In higher dimension, we get a result concerning the first degree Δ^1 under additional assumptions concerning the existence of special Hermitian metrics.

Theorem 2.4. *Let X be a compact complex manifold of complex dimension n endowed with a Hermitian metric g . Suppose that its associated $(1, 1)$ -form ω satisfies one of the following conditions:*

- (a): *either ω^{n-2} is the $(n-2, n-2)$ -component of a d -closed $(2n-4)$ -form;*
- (b): *or $d\omega^{n-2} \in \text{im } d d^c$.*

Then $\Delta^1 = 0$.

Note that the assumptions are trivially satisfied for compact complex surfaces. On the other side, as in [7, 16], we show that 6-dimensional nilmanifolds endowed with left-invariant complex structures never admit Hermitian metrics as above, see Proposition 2.3.

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1. SUMMARY ON NON-KÄHLERNESS DEGREE FOR COMPACT COMPLEX SURFACES

If X is a compact complex surface, then the validity of the $\partial\bar{\partial}$ -Lemma is equivalent to X admitting Kähler metrics. This follows from Kählerness being topologically characterized by b_1 even, [22, 25, 27], see also [23, Corollaire 5.7], and [13, Theorem 11]. In fact, in [29, Lemma 2.3], it is proven that

$$\Delta^1 = 0$$

always for compact complex surfaces. Hence non-Kählerness is measured by just $\Delta^2 \in \mathbb{N}$. Explicit examples are studied in [3], including Inoue and Kodaira surfaces, [3, Theorem 4.1], and class VII surfaces, [3, Theorem 2.2], and in any case it is shown that $\Delta^2 = 2$.

In fact, the following result by A. Teleman holds, see [29, 24].

Theorem 1.1 ([29, Lemma 2.3]). *Let X be a compact complex surface. Then:*

- b_1 is even if and only if $\Delta^1 = \Delta^2 = 0$;
- b_1 is odd if and only if $\Delta^1 = 0$ and $\Delta^2 = 2$.

Proof. For the sake of completeness, we provide a proof that $\Delta^2 \in \{0, 2\}$ according to the parity of b_1 : we are going to extend it to higher dimensions in the next Section.

First, note that the sequence

$$(1) \quad 0 \rightarrow H_{dR}^1(X; \mathbb{C}) \rightarrow H_A^1(X) \xrightarrow{d} H_{BC}^2(X) \rightarrow H_{dR}^2(X; \mathbb{C})$$

is exact. (This holds, in general, on a compact complex manifold X of finite dimension.) Indeed, take $[\alpha] \in H_{dR}^1(X; \mathbb{C})$ such that $\alpha = \partial f + \bar{\partial} g$ for $f, g \in \mathcal{C}^\infty(X; \mathbb{C})$. Since $d\alpha = 0$, then $\partial\bar{\partial}(f - g) = 0$. Therefore $f - g$ is constant, and $\alpha = d f$ gives the zero class in $H_{dR}^1(X; \mathbb{C})$.

Fix a Hermitian metric g being Gauduchon. This means that its associated $(1, 1)$ -form ω satisfies $\partial\bar{\partial}\omega = 0$. It exists by [19, Théorème 1]. Consider the operator

$$\mathcal{D}: \mathcal{C}^\infty(X; \mathbb{C}) \rightarrow \wedge^4 X \otimes \mathbb{C}, \quad \mathcal{D}(f) := d d^c f \wedge \omega.$$

The following sequence is exact:

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{C}^\infty(X; \mathbb{C}) \xrightarrow{\mathcal{D}} \wedge^4 X \otimes \mathbb{C} \xrightarrow{\int_X} \mathbb{C} \rightarrow 0.$$

Indeed, the operator \mathcal{D} is elliptic, with the same symbol as the Laplacian, see [18]. Laplacian is self-adjoint, hence its index is equal to 0. Since the index of an elliptic operator depends only on the symbol, then the index of \mathcal{D} vanishes. However, by Hopf maximum principle, a function in $\ker \mathcal{D}$ which has local maxima is also constant. Therefore, the kernel of \mathcal{D} is one-dimensional. Since the index of \mathcal{D} vanishes, its cokernel is also one-dimensional.

Consider the map

$$\langle - | \omega \rangle : \wedge^2 X \otimes \mathbb{C} \rightarrow \mathbb{C}, \quad \langle \alpha | \omega \rangle := \int_X \alpha \wedge \omega.$$

The sequence

$$(2) \quad 0 \rightarrow H_{dR}^1(X; \mathbb{C}) \rightarrow H_A^1(X) \xrightarrow{\langle - | \omega \rangle^{\text{od}}} \mathbb{C}$$

is exact. Indeed, take $[\alpha] \in H_A^1(X)$ such that $\int_X d\alpha \wedge \omega = 0$. Therefore $d\alpha \wedge \omega \in \ker \int_X - = \text{im } \mathcal{D}$. Let $f \in \mathcal{C}^\infty(X; \mathbb{C})$ be such that $d\alpha \wedge \omega = d d^c f \wedge \omega$. Consider $\tilde{\alpha} := \alpha - d^c f$ and note that $[\tilde{\alpha}] = [\alpha] \in H_A^1(X)$. Then $d\tilde{\alpha} \wedge \omega = 0$. This means that $d\tilde{\alpha}$ is a primitive 2-form, hence anti-self-dual. Since it is also exact, it follows that $d\tilde{\alpha} = 0$, proving the claim.

If b_1 is even, then X admits a Kähler metric, see [23, Corollaire 5.7] and [13, Theorem 11]. Then it follows that $\Delta^1 = \Delta^2 = 0$.

If b_1 is odd, then $\langle - | \omega \rangle \circ d \neq 0$. Indeed, note that $\dim_{\mathbb{C}} H_A^1(X) = 2 \dim_{\mathbb{C}} H_A^{1,0}(X)$ is always even. In particular, we have also

$$H_A^1(X) \xrightarrow{\langle - | \omega \rangle^{\text{od}}} \mathbb{C} \rightarrow 0.$$

Therefore, by (1) and (2),

$$\begin{aligned}\Delta^2 &= \dim_{\mathbb{C}} H_{BC}^2(X) + \dim_{\mathbb{C}} H_A^2(X) - 2b_2 = 2(\dim_{\mathbb{C}} H_{BC}^2(X) - b_2) \\ &\leq 2((\dim_{\mathbb{C}} H_A^1(X) + b_2 - b_1) - b_2) \\ &= 2.\end{aligned}$$

On the other hand, the even number Δ^2 satisfies $\Delta^2 > 0$ by [8, Theorem B]. \square

We notice that this result provides an answer to a question by A. Fujiki, asking whether Δ^2 may change under deformations of the complex structure.

Corollary 1.2. *For compact complex surfaces, Δ^1 and Δ^2 are topological invariants.*

Note that this is no more true in higher-dimension: see the examples on the Iwasawa manifold in [2], and the examples on the Nakamura manifold in [4, 5].

2. NON-KÄHLERNESS 1ST DEGREE FOR HIGHER-DIMENSIONAL COMPLEX MANIFOLDS

Let X be a compact complex manifold of complex dimension n endowed with a Hermitian metric g . Denote by ω its associated $(1,1)$ -form. Recall that

$$\mathcal{D}: \mathcal{C}^\infty(X; \mathbb{C}) \rightarrow \wedge^{2n} X \otimes \mathbb{C}, \quad \mathcal{D}(f) := d d^c f \wedge \omega^{n-1}$$

is an elliptic differential operator with index zero and 1-dimensional kernel, [18].

Define the Hermitian degree

$$\deg: H_{BC}^{1,1}(X) \rightarrow \frac{\wedge^{2n} X \otimes \mathbb{C}}{\text{im } \mathcal{D}} \simeq \mathbb{C}, \quad \deg([\alpha]) := \alpha \wedge \omega^{n-1}.$$

Lemma 2.1. *Let X be a compact complex manifold of complex dimension n endowed with a Hermitian metric g . Suppose that its associated $(1,1)$ -form ω satisfies one of the following conditions:*

- (a): *either ω^{n-2} is the $(n-2, n-2)$ -component of a d -closed $(2n-4)$ -form;*
- (b): *or $d\omega^{n-2} \in \text{im } d d^c$.*

If $[d\alpha] \in H_{BC}^{1,1}(X)$ is such that $\deg([d\alpha]) = 0$, then $[d\alpha] = 0$.

Proof. By the hypothesis: take $f \in \mathcal{C}^\infty(X; \mathbb{C})$ such that

$$d\alpha \wedge \omega^{n-1} = d d^c f \wedge \omega^{n-1}.$$

Set $\alpha' := \alpha - d^c f$. Note that $[d\alpha] = [d\alpha']$ in $H_{BC}^{1,1}(X)$, and that $d\alpha'$ is primitive. Hence use Weyl identity to write $*\alpha' = \frac{1}{(n-2)!} \alpha' \wedge \omega^{n-1}$.

In case (a), take $\beta \in \wedge^{2n-4}$ such that

$$\omega^{n-2} = \pi_{\wedge^{n-2, n-2} X} \beta \quad \text{with} \quad d\beta = 0.$$

(Here $\pi_{\wedge^{n-2, n-2} X}$ denotes the natural projections onto $\wedge^{n-2, n-2} X$.) We have:

$$\begin{aligned}\|d\alpha'\|^2 &= \int_X d\alpha' \wedge *(d\alpha') = \frac{1}{(n-2)!} \int_X d\alpha' \wedge d\alpha' \wedge \omega^{n-2} \\ &= \frac{1}{(n-2)!} \int_X d\alpha' \wedge d\alpha' \wedge \pi_{\wedge^{n-2, n-2} X} \beta \\ &= \frac{1}{(n-2)!} \int_X d\alpha' \wedge d\alpha' \wedge \beta \\ &= -\frac{1}{(n-2)!} \int_X \alpha' \wedge d\alpha' \wedge d\beta = 0,\end{aligned}$$

thus giving $d\alpha' = 0$.

In case (b), take $\eta \in \wedge^{n-2, n-3} X \oplus \wedge^{n-3, n-2} X$ such that

$$d\omega^{n-2} = dd^c \eta.$$

$$\begin{aligned} \|d\alpha'\|^2 &= \int_X d\alpha' \wedge *(d\alpha') = \frac{1}{(n-2)!} \int_X d\alpha' \wedge d\alpha' \wedge \omega^{n-2} \\ &= -\frac{1}{(n-2)!} \int_X \alpha' \wedge d\alpha' \wedge d\omega^{n-2} = -\frac{1}{(n-2)!} \int_X \alpha' \wedge d\alpha' \wedge dd^c \eta \\ &= \frac{1}{(n-2)!} \int_X d\alpha' \wedge d\alpha' \wedge d^c \eta = -\frac{2}{(n-2)!} \int_X d^c d\alpha' \wedge d\alpha' \wedge \eta = 0, \end{aligned}$$

thus giving $d\alpha' = 0$. \square

Remark 2.2. Note that both the conditions (a) and (b) in Lemma 2.1 yields that $dd^c \omega^{n-2} = 0$. That is, ω is *astheno-Kähler* in the sense of J. Jost and S.-T. Yau, [21]. Note that the conditions are trivially satisfied in case $2n = 4$. In a sense, condition (a) is the $(n-2)$ -degree counterpart of Hermitian-symplectic condition in the sense of [28]. Note that a Hermitian metric satisfying $d\omega^{n-2} = 0$ is actually Kähler, [20].

The following generalizes the result in [7], see also [16], in proving that conditions (a) and (b) are not satisfied for 6-dimensional nilmanifolds with invariant structures.

Proposition 2.3. *On a 6-dimensional non-torus nilmanifold endowed with a left-invariant complex structure, there is no left-invariant metric with either the property (a) or (b) in Lemma 2.1.*

Proof. In dimension 6, condition (a) is equivalent to having a Hermitian-symplectic structure. This is proven to be impossible on 6-dimensional nilmanifold in [7, Theorem 3.3], except the torus; see also [16, Theorem 1.3].

We prove that condition (b) is impossible, too. In fact, let $X = \Gamma \backslash G$ be a 6-dimensional nilmanifold endowed with a G -left-invariant complex structure. Up to symmetrization, [10, Theorem 7], we can suppose that there exists a G -left-invariant metric with associated $(1,1)$ -form Ω such that $d\Omega = dd^c \eta$ where η is a G -left-invariant 1-form. In particular, such a metric is also strong Kähler with torsion. Therefore, by [17, Theorem 1.2], take a G -left-invariant co-frame $\{\varphi^1, \varphi^2, \varphi^3\}$ of $(T^{1,0}X)^*$ with structure equations

$$\begin{cases} d\varphi^1 &= 0 \\ d\varphi^2 &= 0 \\ d\varphi^3 &= A\varphi^{\bar{1}\bar{2}} + B\varphi^{\bar{2}\bar{2}} + C\varphi^{1\bar{1}} + D\varphi^{1\bar{2}} + E\varphi^{12} \end{cases}$$

where $A, B, C, D, E \in \mathbb{C}$ satisfy $|A|^2 + |D|^2 + |E|^2 + 2\Re(\bar{B}C) = 0$; (we are shortening, e.g., $\varphi^{\bar{1}\bar{2}} := \bar{\varphi}^1 \wedge \varphi^2$). Therefore $dd^c \eta = 0$. That is Ω is the associated $(1,1)$ -form of a Kähler metric. By [11, Theorem A], then X is a torus. \square

By the previous Lemma, we get the following result.

Theorem 2.4. *Let X be a compact complex manifold of complex dimension n endowed with a Hermitian metric g . Suppose that its associated $(1,1)$ -form ω satisfies either condition (a) or (b) in Lemma 2.1. Then $\Delta^1 = 0$.*

Proof. The sequence

$$0 \rightarrow H_{dR}^1(X; \mathbb{C}) \rightarrow H_A^1(X) \xrightarrow{\deg \circ d} \mathbb{C}$$

is exact. Indeed, take $[\alpha]_{dR} \in H_{dR}^1(X; \mathbb{C})$ yielding a zero class in Aeppli cohomology, that is, $\alpha = \partial f + \bar{\partial} g$ for $f, g \in \mathcal{C}^\infty(X; \mathbb{C})$. Since $d\alpha = 0$, then $\partial\bar{\partial}(f - g) = 0$. By the maximum principle, we get $f - g$ constant. Then $\alpha = d f$ yields the zero class in de Rham cohomology. By Lemma 2.1, if $[\alpha]_A \in H_A^1(X)$ satisfies $\deg[d\alpha]_{BC} = 0$, then $[d\alpha]_{BC} = 0$ in $H_{BC}^2(X)$. Hence $d\alpha = d d^c f$ for some $f \in \mathcal{C}^\infty(X; \mathbb{C})$, that is, $d(\alpha - d^c f) = 0$, where $[\alpha]_A = [\alpha - d^c f]_A$.

It follows that

$$\dim_{\mathbb{C}} H_A^1(X) - b_1 = 1 - \dim_{\mathbb{C}} \operatorname{coker}(\deg \circ d) \in \{0, 1\}$$

according to the parity of b_1 . Indeed, $\dim_{\mathbb{C}} H_A^1(X)$ is always even.

Note that the natural map $H_{BC}^1(X) \rightarrow H_{dR}^1(X; \mathbb{C})$ is always injective. Then

$$\begin{aligned} 0 \leq \Delta^1 &= \dim_{\mathbb{C}} H_A^1(X) + \dim_{\mathbb{C}} H_{BC}^1(X) - 2b_1 \\ &\leq (1 - \dim_{\mathbb{C}} \operatorname{coker}(\deg \circ d)) \leq 1. \end{aligned}$$

Since Δ^1 has to be even, this yields $\Delta^1 = 0$. □

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